3.8 — Polynomial Regression

ECON 480 • Econometrics • Fall 2020

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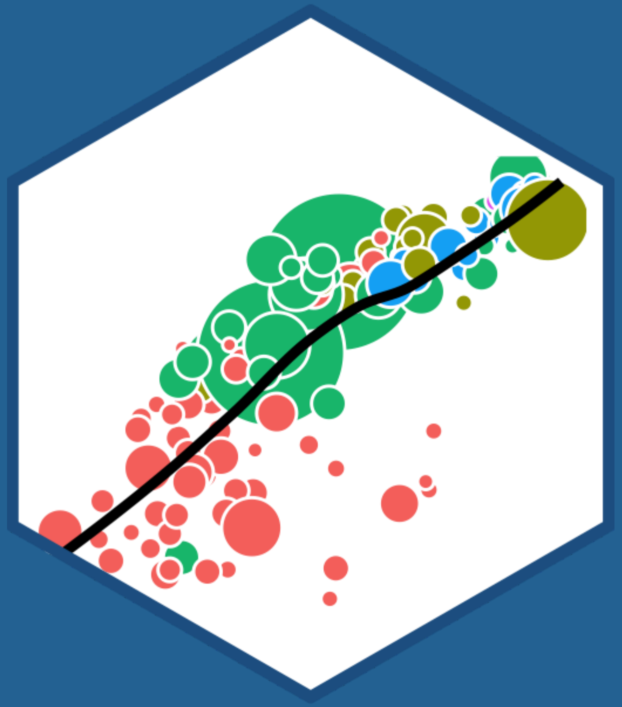
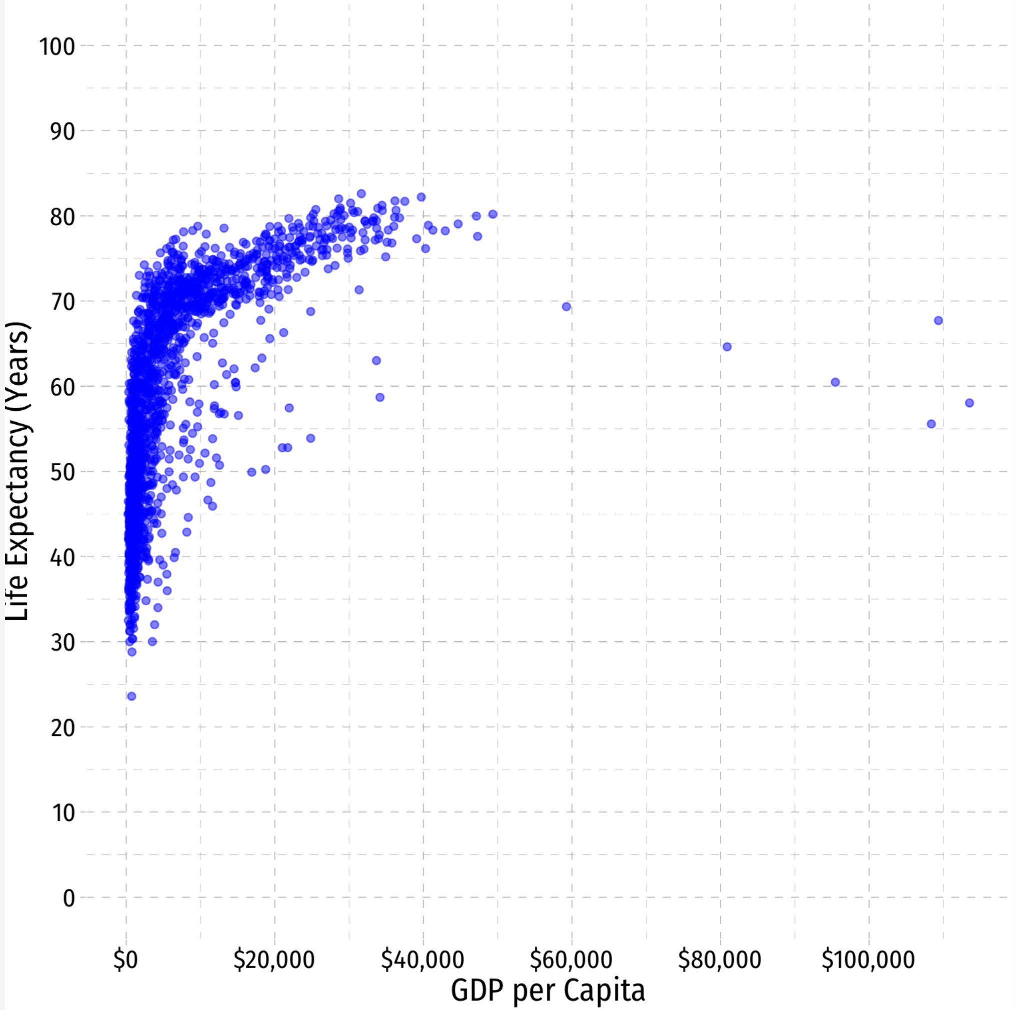


Outline

Linear Regression

- OLS is commonly known as "***linear* regression**" as it fits a *straight line* to data points

- Often, data and relationships between variables may *not* be linear



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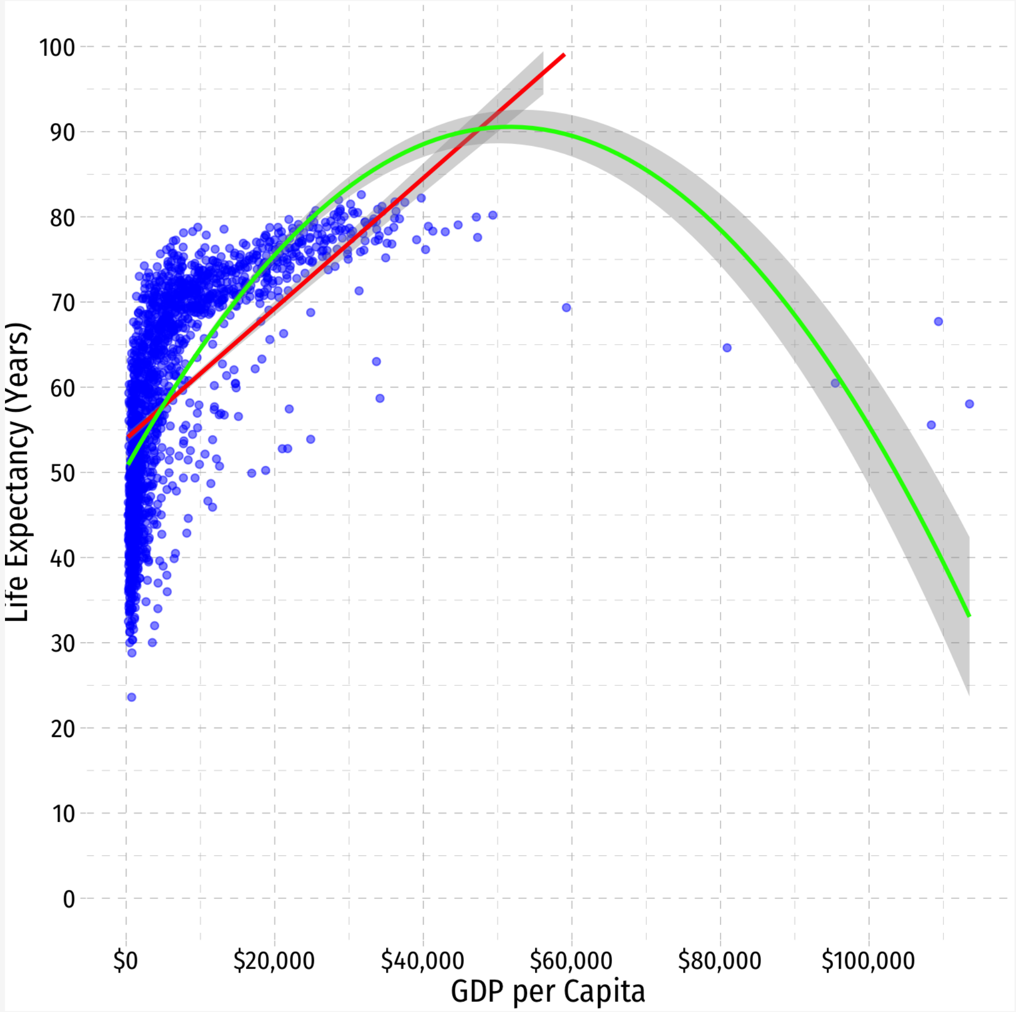

Linear Regression

- OLS is commonly known as "***linear* regression**" as it fits a *straight line* to data points
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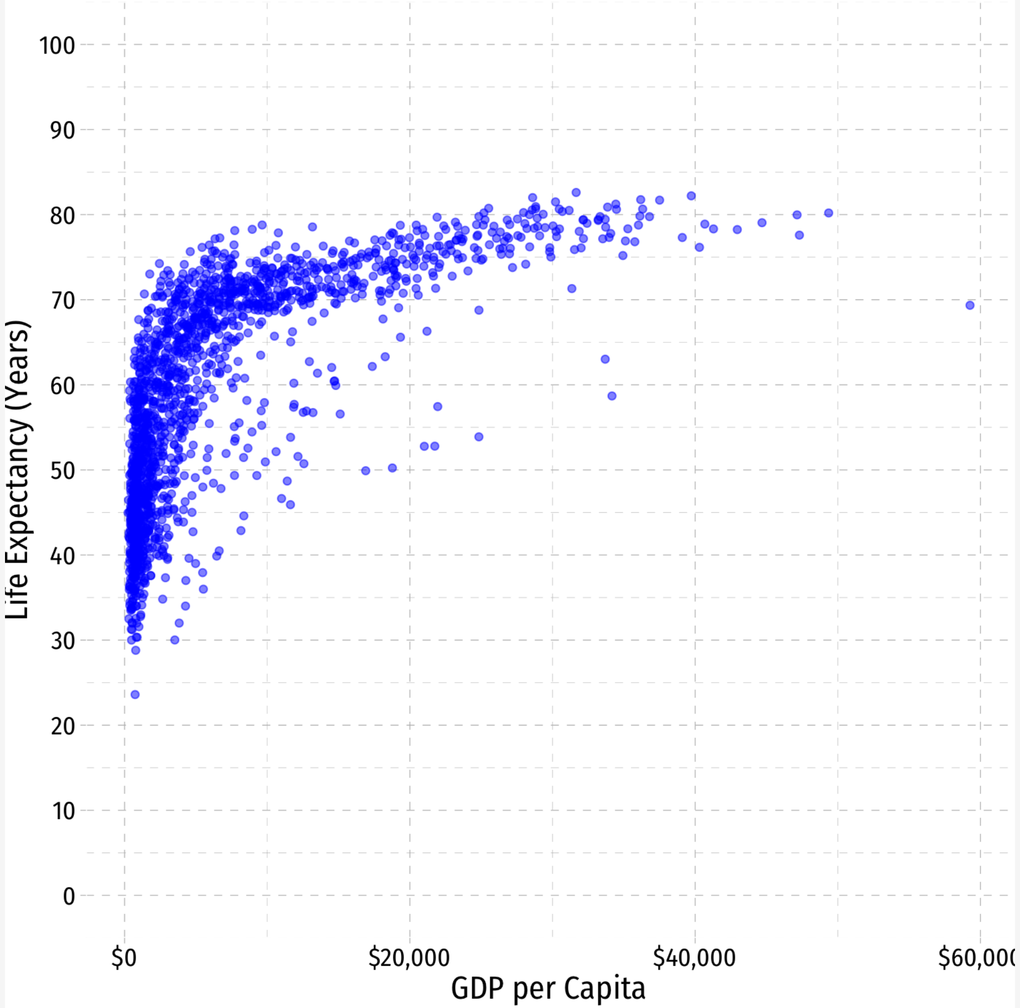
- Get rid of the outliers (>$60,000)



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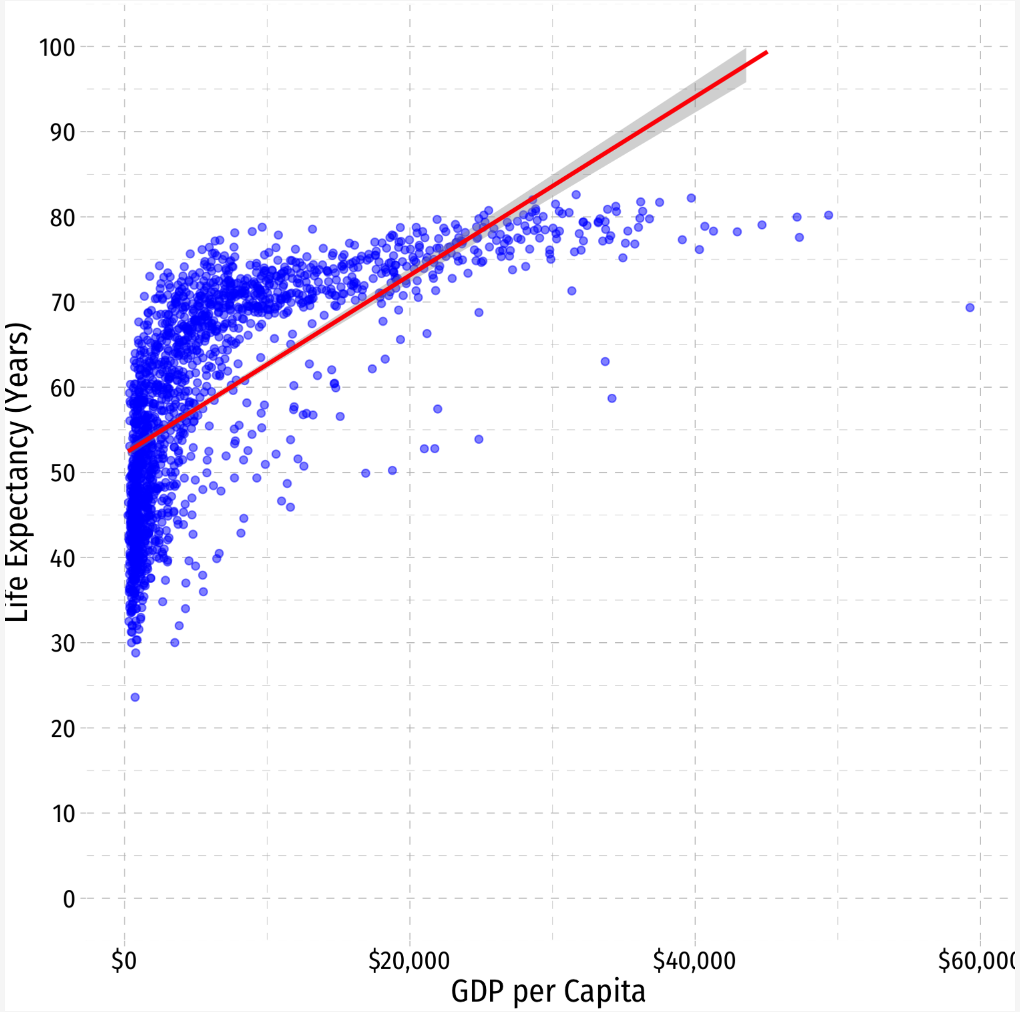

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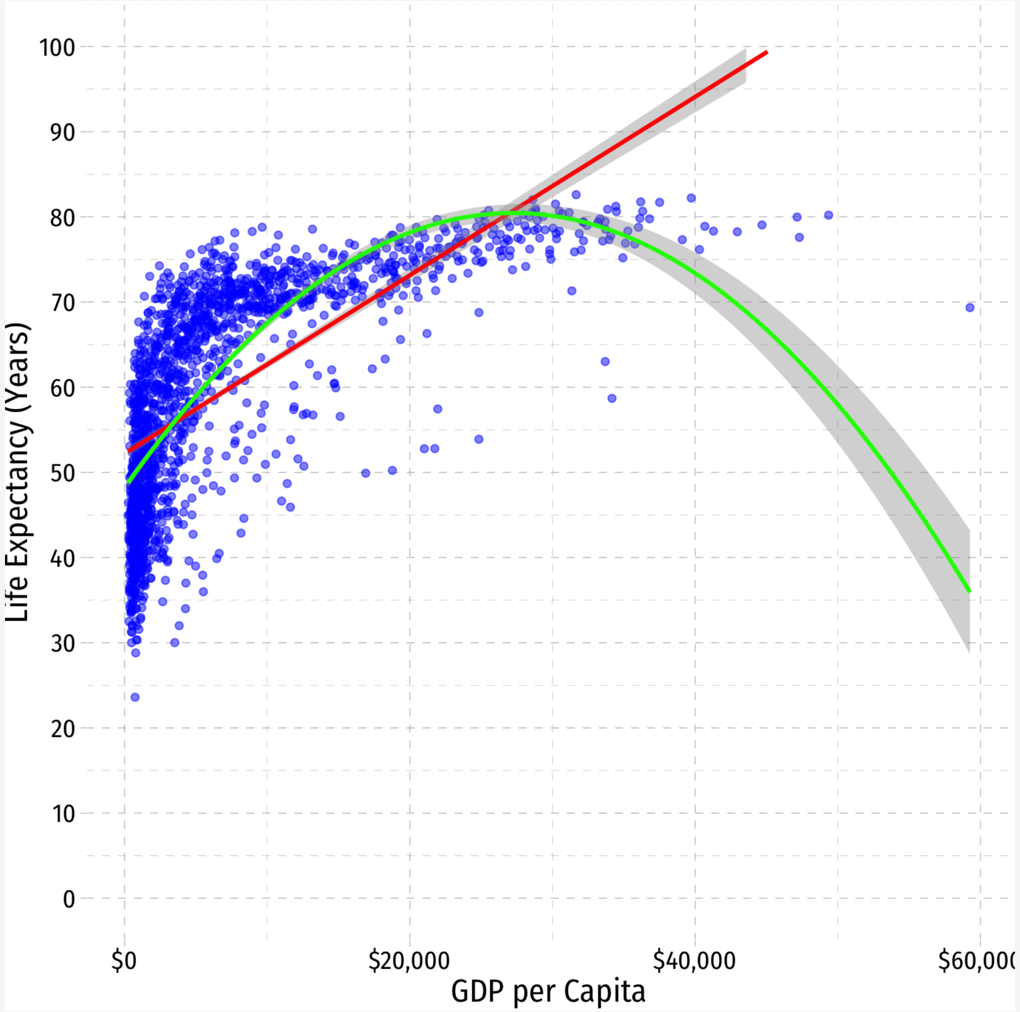

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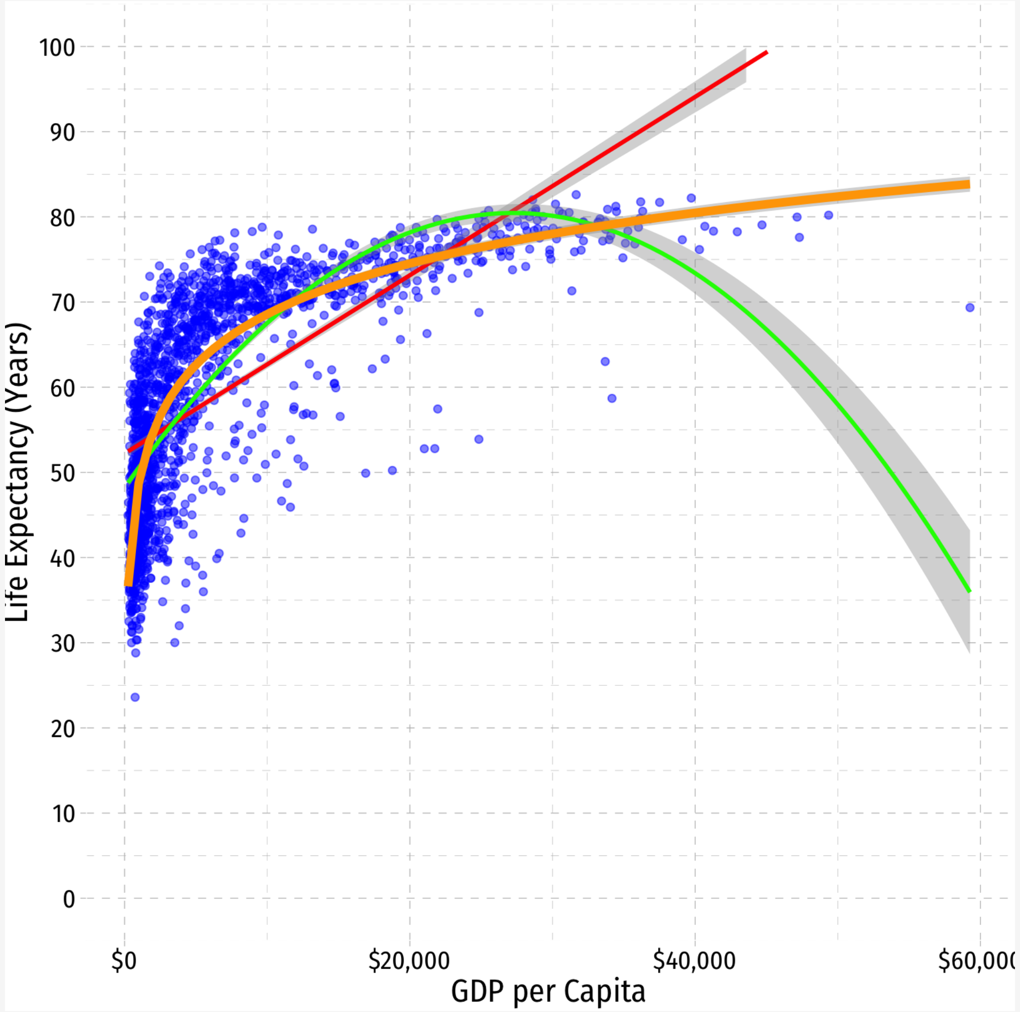


Nonlinear Effects in Linear Regression

- Despite being "linear regression", OLS can handle this with an easy fix

- OLS requires all *parameters* (i.e. the β's) to be linear, the *regressors* (X's) can be nonlinear:

$$Y_i=\beta_0+\beta_1 X_i^2 \quad ✅$$

$$Y_i=\beta_0+\beta_1^2X_i \quad ❌$$

$$Y_i=\beta_0+\beta_1 \sqrt{X_i} \quad ✅$$

$$Y_i=\beta_0+\sqrt{\beta_1} X_i \quad ❌$$

$$Y_i=\beta_0+\beta_1 (X_{1i} \times X_{2i}) \quad ✅$$

$$Y_i=\beta_0+\beta_1 ln(X_i) \quad ✅$$

$$Y_i=\beta_0-e^{\beta_1X_i} \quad ✅$$

- In the end, each X is always just a number in the data, OLS can always estimate parameters for it

Sources of Nonlinearities

- Effect of $X_1 \rightarrow Y$ might be nonlinear if:

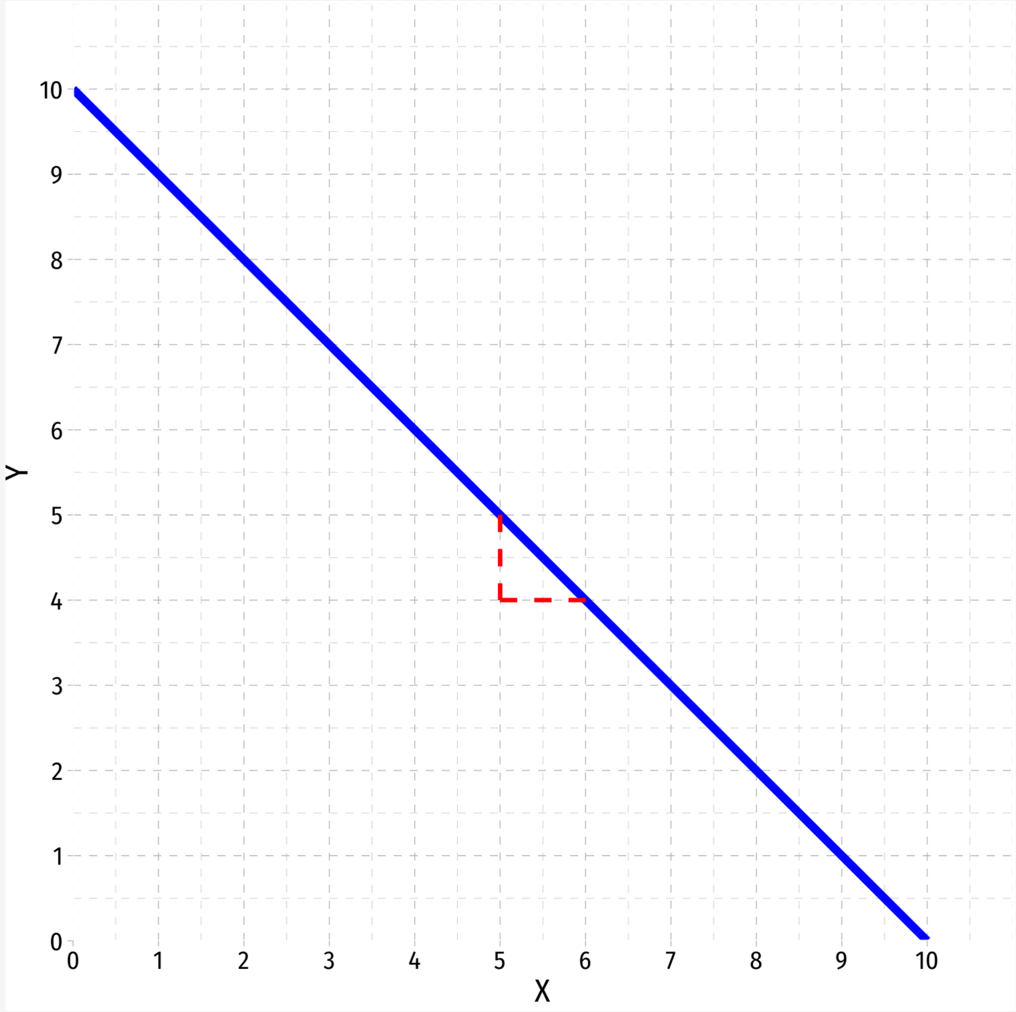
1. $X_1 \rightarrow Y$ is different for different levels of X_1
    - e.g. **diminishing returns**: $\uparrow X_1$ increases Y at a *decreasing* rate
    - e.g. **increasing returns**: $\uparrow X_1$ increases Y at an *increasing* rate

2. $X_1 \rightarrow Y$ is different for different levels of X_2
    - e.g. interaction effects (last lesson)

Nonlinearities Alter Marginal Effects

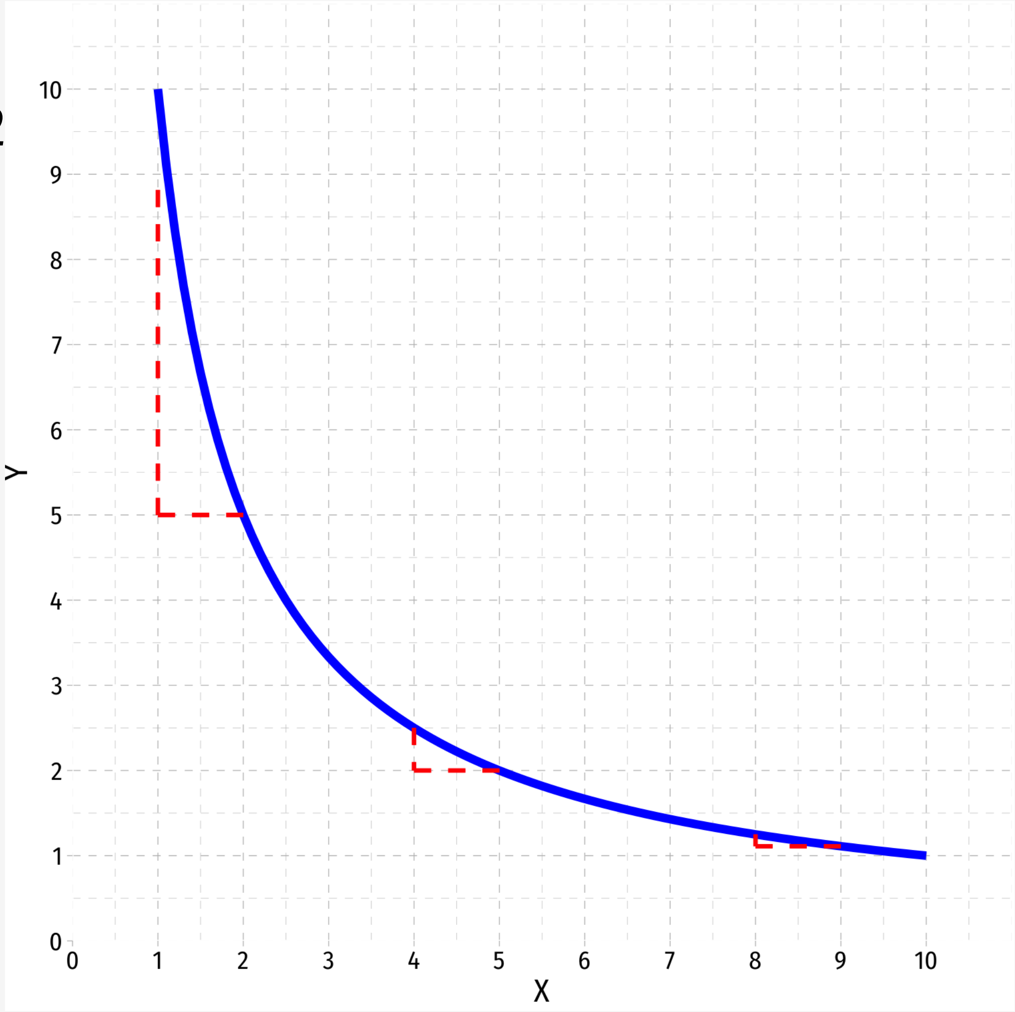
- **Linear**:
  $$Y=\hat{\beta_0}+\hat{\beta_1}X$$

- marginal effect (slope), $(\hat{\beta_1}) = \frac{\Delta Y}{\Delta X}$ is constant for all X



Nonlinearities Alter Marginal Effects

- **Polynomial**:
  $$Y=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2
- Marginal effect, “slope” \(\left(\neq \hat{\beta_1}\right)\) *depends on the value of* \(X\)!



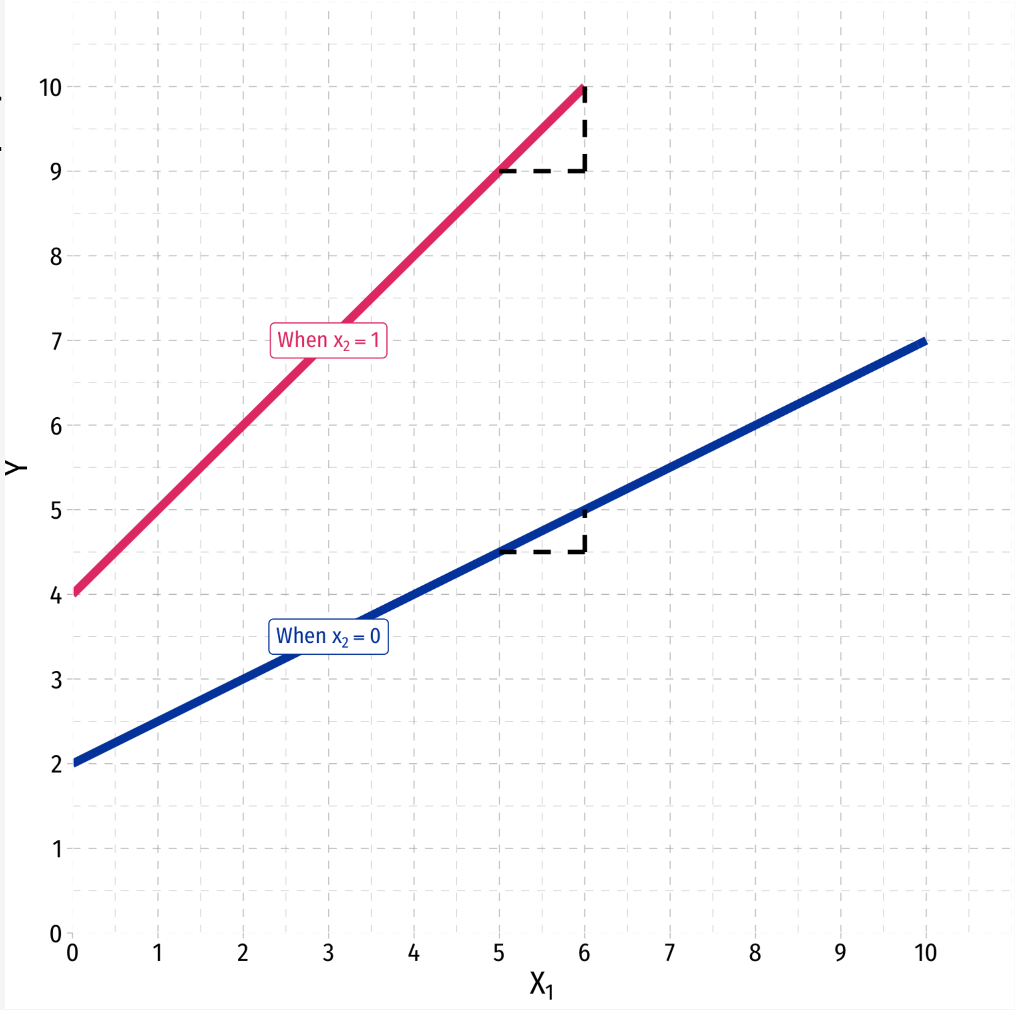
Sources of Nonlinearities III

- **Interaction Effect**: $$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X_1+\hat{
  \times X_2$$

- Marginal effect, “slope” *depends on the value of* \(X_2\)!

- Easy example: if \(X_2\) is a dummy variable:

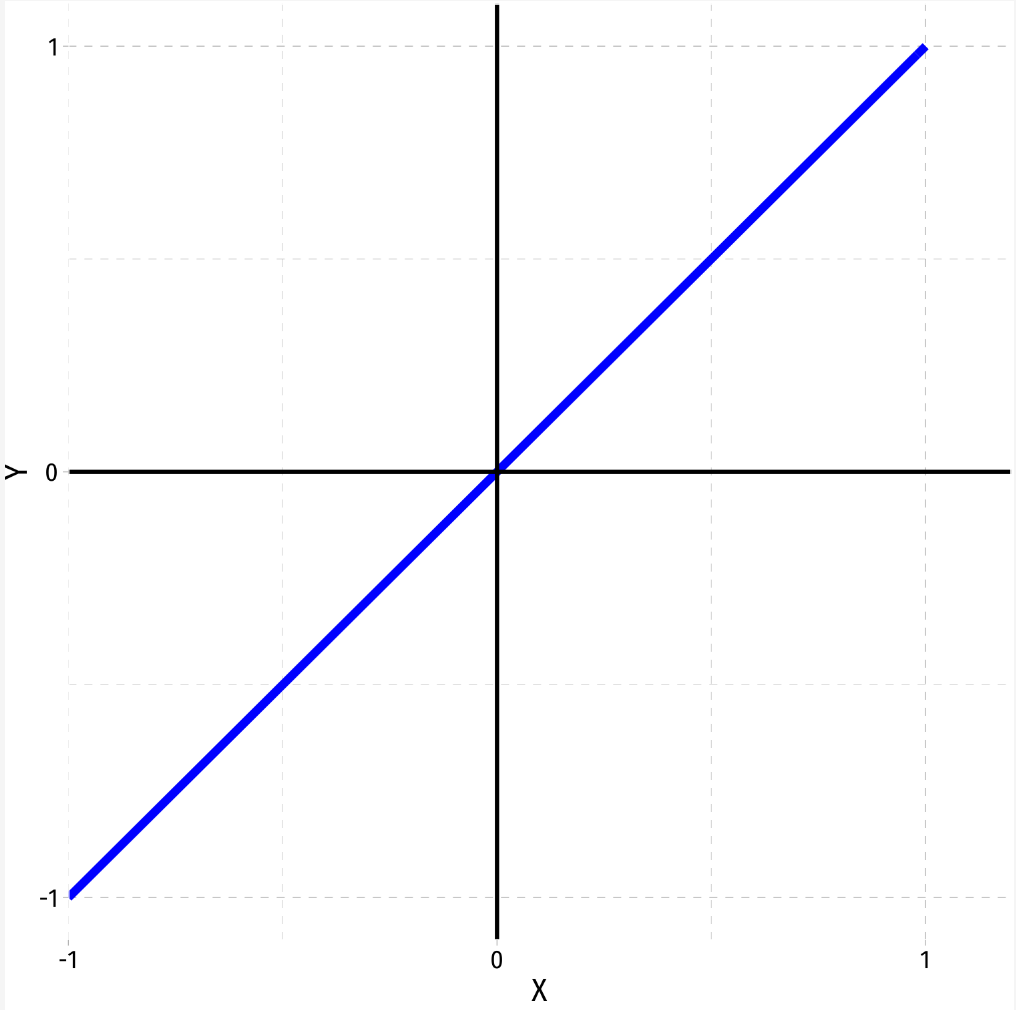
  - \(X_2=0\) (control) vs. \(X_2=1\) (treatment)



Polynomial Functions of \(X\) I

- Linear

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X$$



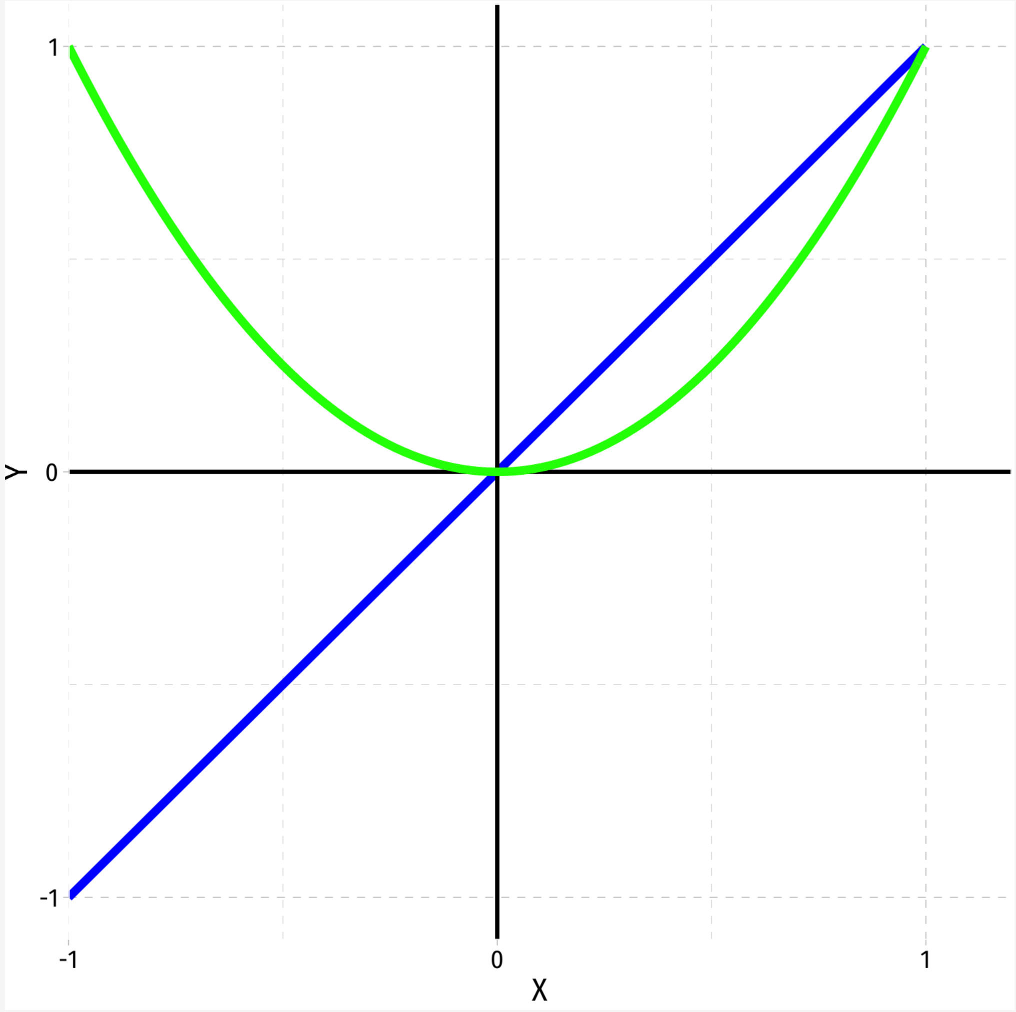
Polynomial Functions of \(X\) I

- Linear

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X$$

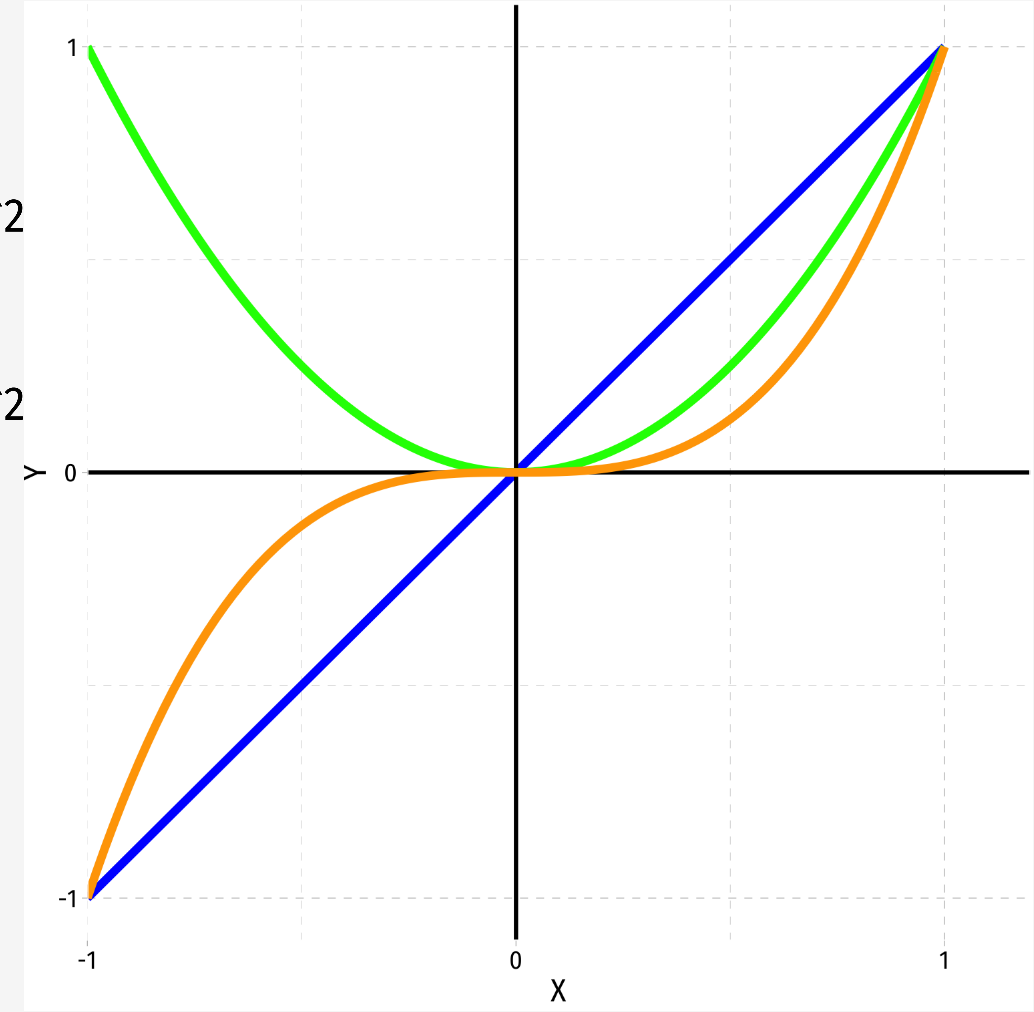
- Quadratic

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2$$



Polynomial Functions of \(X\) I

- Linear, \(\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X\)
- Quadratic, \
(\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2
- Cubic, \
(\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2



Polynomial Functions of \(X\) I

- Linear

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X$$

- Quadratic

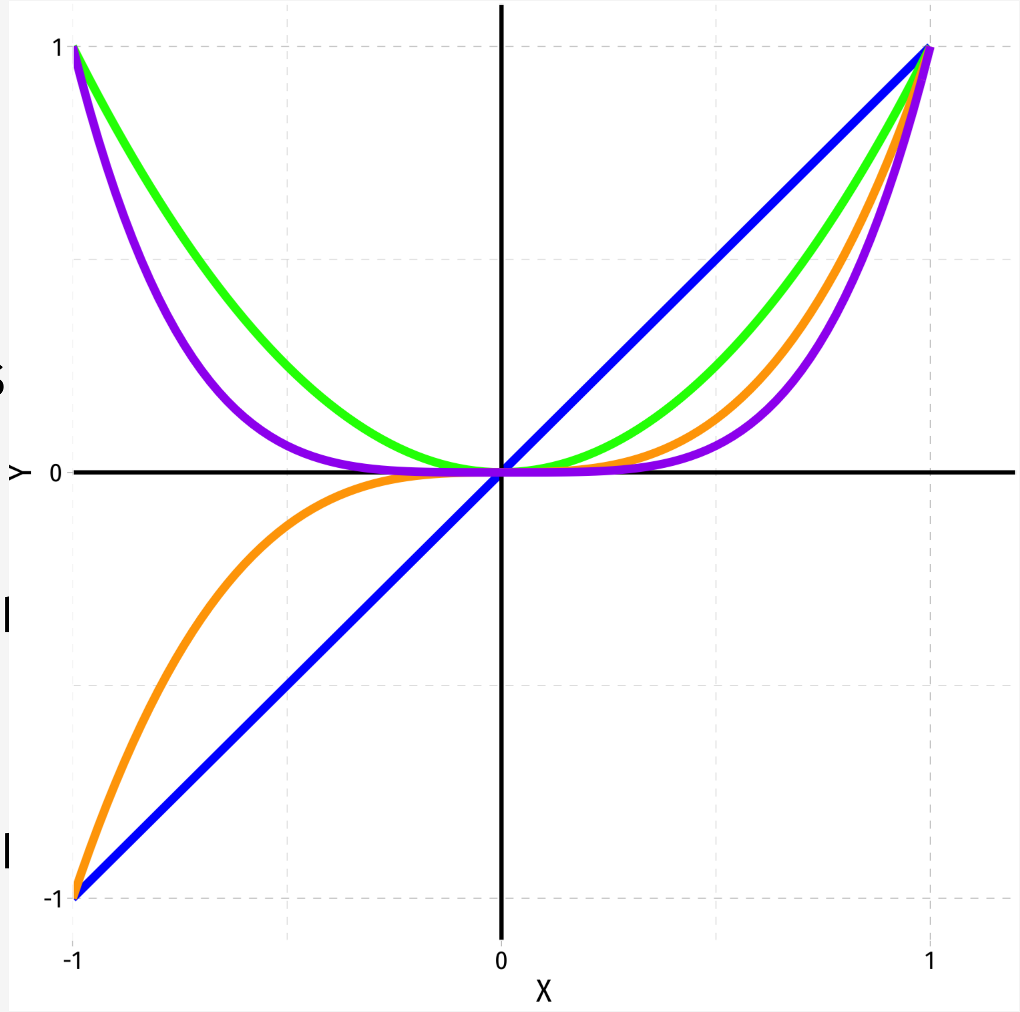
$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2$$

- Cubic

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2+\

- Quartic

$$\hat{Y}=\hat{\beta_0}+\hat{\beta_1}X+\hat{\beta_2}X^2+\



Polynomial Functions of X I

$$\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1} X_i + \hat{\beta_2} X_i^2 + \cdots + \hat{\beta_{\color{#e64173}{r}}} X_i^{\color{#e64173}{r}} + u_i$$

- Where $\color{e64173}{r}$ is the highest power X_i is raised to
    - quadratic $\color{e64173}{r=2}$
    - cubic $\color{e64173}{r=3}$
- The graph of an r^{th}-degree polynomial function has $(r-1)$ bends
- Just another multivariate OLS regression model!

The Quadratic Model

Quadratic Model

$$\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1} X_i + \hat{\beta_2} X_i^2$$

- **Quadratic model** has X and X^2 variables in it (yes, need both!)

- How to interpret coefficients (betas)?

  - β_0 as "intercept" and β_1 as "slope" makes no sense 🧐
  - β_1 as effect $X_i \rightarrow Y_i$ holding X_i^2 constant??[†]

- **Estimate marginal effects** by calculating predicted $\hat{Y_i}$ for different levels of X_i

[†] Note: this is *not* a perfect multicollinearity problem! Correlation only measures *linear* relationships!

Quadratic Model: Calculating Marginal Effects

$$\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1} X_i + \hat{\beta_2} X_i^2$$

- What is the **marginal effect** of \(\Delta X_i \rightarrow \Delta Y_i\)?

- Take the **derivative** of \(Y_i\) with respect to \(X_i\): $$\frac{\partial \, Y_i}{\partial \, X_i} = \hat{\beta_1}+2\hat{\beta_2} X_i$$

- **Marginal effect** of a 1 unit change in \(X_i\) is a \(\color{#6A5ACD}{\left(\hat{\beta_1}+2\hat{\beta_2} X_i \right)}\) unit change in \(Y\)

Quadratic Model: Example I

Example: $$\widehat{\text{Life Expectancy}_i} = \hat{\beta_0}+\hat{\beta_1} \, \text{GDP per capita}_i+\hat{\beta_2}\, \text{GDP per capita}^2_i$$

- Use `gapminder` package and data

```r
library(gapminder)
```

Quadratic Model: Example II

- These coefficients will be very large, so let's transform `gdpPercap` to be in $1,000's

```
gapminder <- gapminder %>%
  mutate(GDP_t = gdpPercap/1000)

gapminder %>% head() # look at it
```

country	continent	year	lifeExp	pop	gdpPercap	GDP_t
<fctr>	<fctr>	<int>	<dbl>	<int>	<dbl>	<dbl>
Afghanistan	Asia	1952	28.801	8425333	779.4453	0.7794453
Afghanistan	Asia	1957	30.332	9240934	820.8530	0.8208530
Afghanistan	Asia	1962	31.997	10267083	853.1007	0.8531007
Afghanistan	Asia	1967	34.020	11537966	836.1971	0.8361971
Afghanistan	Asia	1972	36.088	13079460	739.9811	0.7399811
Afghanistan	Asia	1977	38.438	14880372	786.1134	0.7861134

6 rows

Quadratic Model: Example III

- Let's also create a squared term, `gdp_sq`

```
gapminder <- gapminder %>%
  mutate(GDP_sq = GDP_t^2)

gapminder %>% head() # look at it
```

country <fctr>	continent <fctr>	year <int>	lifeExp <dbl>	pop <int>	gdpPercap <dbl>	GDP_t <dbl>	GDP_sq <dbl>
Afghanistan	Asia	1952	28.801	8425333	779.4453	0.7794453	0.6075350
Afghanistan	Asia	1957	30.332	9240934	820.8530	0.8208530	0.6737997
Afghanistan	Asia	1962	31.997	10267083	853.1007	0.8531007	0.7277808
Afghanistan	Asia	1967	34.020	11537966	836.1971	0.8361971	0.6992257
Afghanistan	Asia	1972	36.088	13079460	739.9811	0.7399811	0.5475720
Afghanistan	Asia	1977	38.438	14880372	786.1134	0.7861134	0.6179742

6 rows

Quadratic Model: Example IV

- Can "manually" run a multivariate regression with `GDP_t` and `GDP_sq`

```r
library(broom)
reg1<-lm(lifeExp ~ GDP_t + GDP_sq, data = gapminder)

reg1 %>% tidy()
```

term <chr>	estimate <dbl>	std.error <dbl>	statistic <dbl>	p.value <dbl>
(Intercept)	50.52400578	0.2978134673	169.64984	0.000000e+00
GDP_t	1.55099112	0.0373734945	41.49976	1.292863e-260
GDP_sq	-0.01501927	0.0005794139	-25.92149	3.935809e-125

3 rows

Quadratic Model: Example V

- OR use `gdp_t` and add the "transform" command in regression, `I(gdp_t^2)`

```
reg1_alt<-lm(lifeExp ~ GDP_t + I(GDP_t^2), data = gapminder)

reg1_alt %>% tidy()
```

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	50.52400578	0.2978134673	169.64984	0.000000e+00
GDP_t	1.55099112	0.0373734945	41.49976	1.292863e-260
I(GDP_t^2)	-0.01501927	0.0005794139	-25.92149	3.935809e-125

3 rows

Quadratic Model: Example VI

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	50.52400578	0.2978134673	169.64984	0.000000e+00
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GDP_sq	-0.01501927	0.0005794139	-25.92149	3.935809e-125

3 rows

$$\widehat{\text{Life Expectancy}_i} = 50.52+1.55 \, \text{GDP per capita}_i - 0.02\, \text{GDP per capita}^2_i$$

- Positive effect $(\hat{\beta_1}>0)$, with diminishing returns $(\hat{\beta_2}<0)$
- Effect on Life Expectancy of increasing GDP depends on initial value of GDP!

Quadratic Model: Example VII

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	50.52400578	0.2978134673	169.64984	0.000000e+00
GDP_t	1.55099112	0.0373734945	41.49976	1.292863e-260
GDP_sq	-0.01501927	0.0005794139	-25.92149	3.935809e-125

3 rows

- **Marginal effect** of GDP per capita on Life Expectancy:

$$\begin{align*} \frac{\partial \, Y}{\partial \; X} &= \hat{\beta_1}+2\hat{\beta_2} X_i\\ \frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} &\approx 1.55+2(-0.02) \, \text{GDP}\\ &\approx \color{#e64173}{1.55-0.04 \, \text{GDP}}\\ \end{align*}$$

Quadratic Model: Example VIII

$$\frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} = 1.55-0.04 \, \text{GDP}$$

Marginal effect of GDP if GDP $=5$ ($ thousand):

$$\begin{align*} \frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} &= 1.55-0.04\text{GDP}\\ &= 1.55-0.04(5)\\ &= 1.55-0.20\\ &=1.35\\ \end{align*}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy increases by 1.35 years

Quadratic Model: Example IX

$$\frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} = 1.55-0.04 \, \text{GDP}$$

Marginal effect of GDP if GDP $=25$ ($ thousand):

$$\begin{align*} \frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} &= 1.55-0.04\text{GDP}\\ &= 1.55-0.04(25)\\ &= 1.55-1.00\\ &=0.55\\ \end{align*}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy increases by 0.55 years

Quadratic Model: Example X

$$\frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} = 1.55-0.04 \, \text{GDP}$$

Marginal effect of GDP if GDP $=50$ ($ thousand):

$$\begin{align*} \frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} &= 1.55-0.04\text{GDP}\\ &= 1.55-0.04(50)\\ &= 1.55-2.00\\ &=-0.45\\ \end{align*}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy *decreases* by 0.45 years

Quadratic Model: Example XI

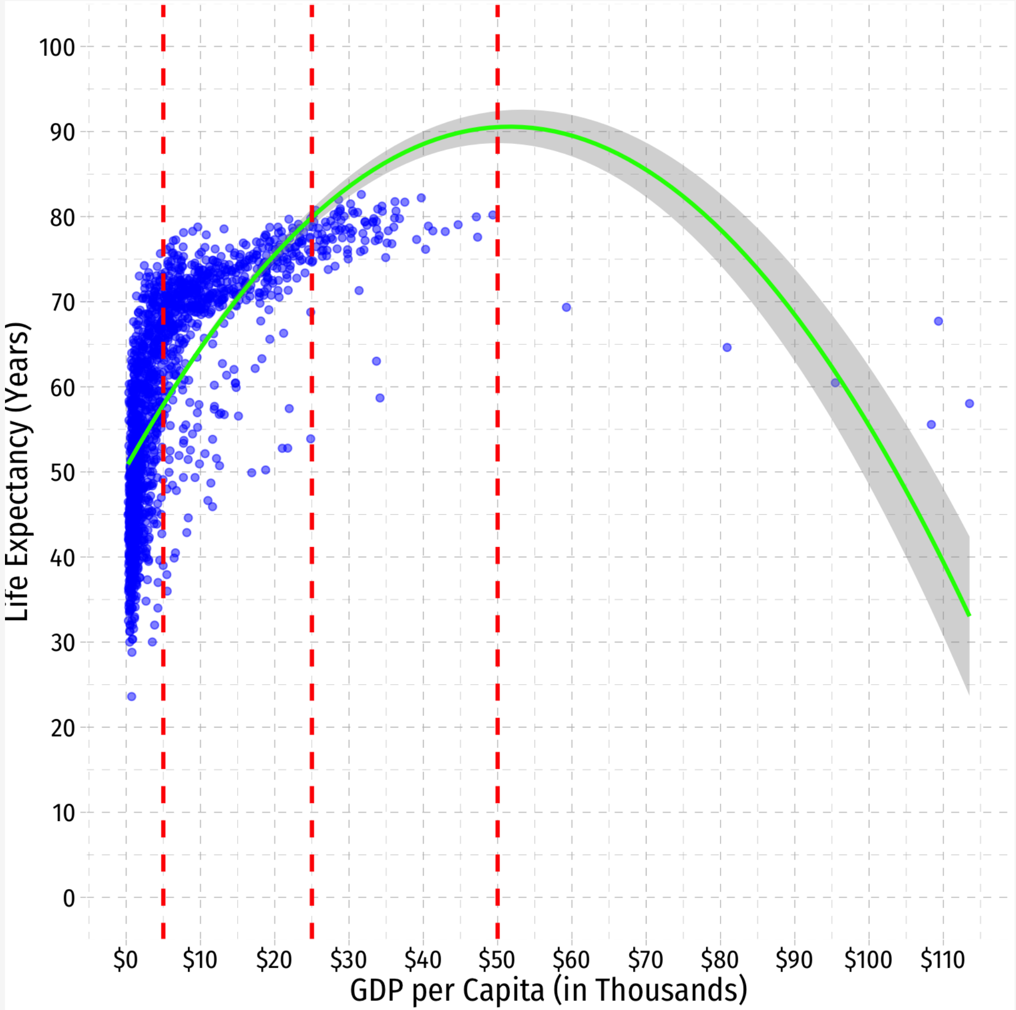
$$\begin{align*}\widehat{\text{Life Expectancy}_i} &= 50.52+1.55 \, \text{GDP per capita}_i - 0.02\, \text{GDP per capita}^2_i \\ \frac{\partial \, \text{Life Expectancy}}{d \, \text{GDP}} &= 1.55-0.04\text{GDP} \\ \end{align*}$$

***Initial* GDP per capita**	**Marginal Effect†**
$5,000	1.35 years
$25,000	0.55 years
$50,000	-0.45 years

† Of +$1,000 GDP/capita on Life Expectancy.

Quadratic Model: Example XII

```
ggplot(data = gapminder)+
  aes(x = GDP_t,
      y = lifeExp)+
  geom_point(color="blue", alpha=0.5)+
  stat_smooth(method = "lm",
              formula = y ~ x + I(x^2),
              color="green")+
  geom_vline(xintercept=c(5,25,50),
             linetype="dashed",
             color="red", size = 1)+
  scale_x_continuous(labels=scales::dollar,
                     breaks=seq(0,120,10))+
  scale_y_continuous(breaks=seq(0,100,10),
                     limits=c(0,100))+
  labs(x = "GDP per Capita (in Thousands)",
       y = "Life Expectancy (Years)")+
  ggthemes::theme_pander(base_family = "Fira Sans Condensed",
           base_size=16)
```



The Quadratic Model: Maxima and Minima

Quadratic Model: Maxima and Minima I

- For a polynomial model, we can also find the predicted **maximum** or **minimum** of $\hat{Y_i}$

- A quadratic model has a single global maximum or minimum (1 bend)

- By calculus, a minimum or maximum occurs where:

$$\begin{align*} \frac{ \partial \, Y_i}{\partial \, X_i} &=0\\ \beta_1 + 2\beta_2 X_i &= 0\\ 2\beta_2 X_i&= -\beta_1\\ X_i^*&=-\frac{\beta_1}{2\beta_2}\\ \end{align*}$$

Quadratic Model: Maxima and Minima II

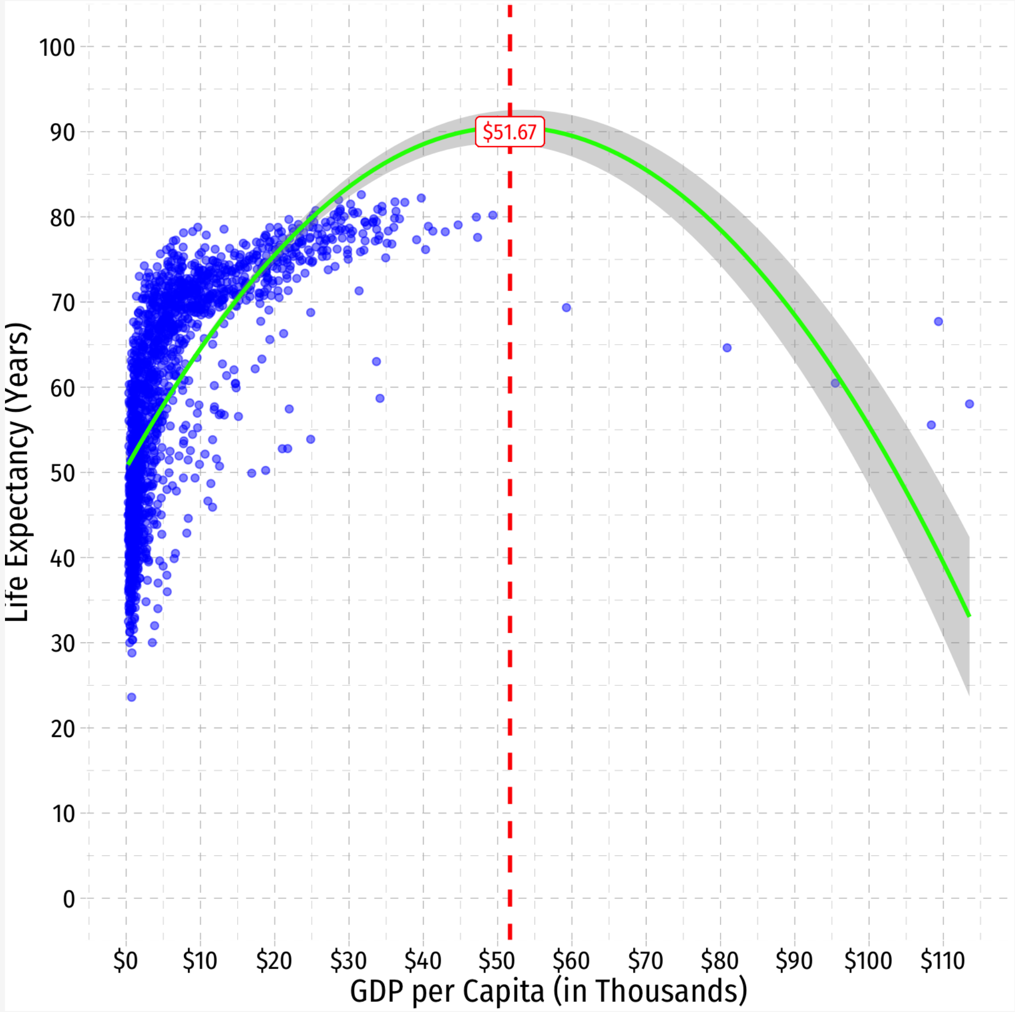
term	estimate	std.error	statistic
<chr>	<dbl>	<dbl>	<dbl>
(Intercept)	50.52400578	0.2978134673	169.64984
GDP_t	1.55099112	0.0373734945	41.49976
GDP_sq	-0.01501927	0.0005794139	-25.92149

3 rows | 1-4 of 5 columns

$$\begin{align*} GDP_i^*&=-\frac{\beta_1}{2\beta_2}\\ GDP_i^*&=-\frac{(1.55)}{2(-0.015)}\\ GDP_i^*& \approx 51.67\\ \end{align*}$$

Quadratic Model: Maxima and Minima III

```
ggplot(data = gapminder)+
  aes(x = GDP_t,
      y = lifeExp)+
  geom_point(color="blue", alpha=0.5)+
  stat_smooth(method = "lm",
              formula = y ~ x + I(x^2),
              color="green")+
  geom_vline(xintercept=51.67, linetype="dashed", color="red"
  geom_label(x=51.67, y=90, label="$51.67", color="red")+
  scale_x_continuous(labels=scales::dollar,
                     breaks=seq(0,120,10))+
  scale_y_continuous(breaks=seq(0,100,10),
                     limits=c(0,100))+
  labs(x = "GDP per Capita (in Thousands)",
       y = "Life Expectancy (Years)")+
  ggthemes::theme_pander(base_family = "Fira Sans Condensed",
           base_size=16)
```



Are Polynomials Necessary?

Determining Polynomials are Necessary I

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	50.52400578	0.2978134673	169.64984	0.000000e+00
GDP_t	1.55099112	0.0373734945	41.49976	1.292863e-260
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3 rows

- Is the quadratic term necessary?

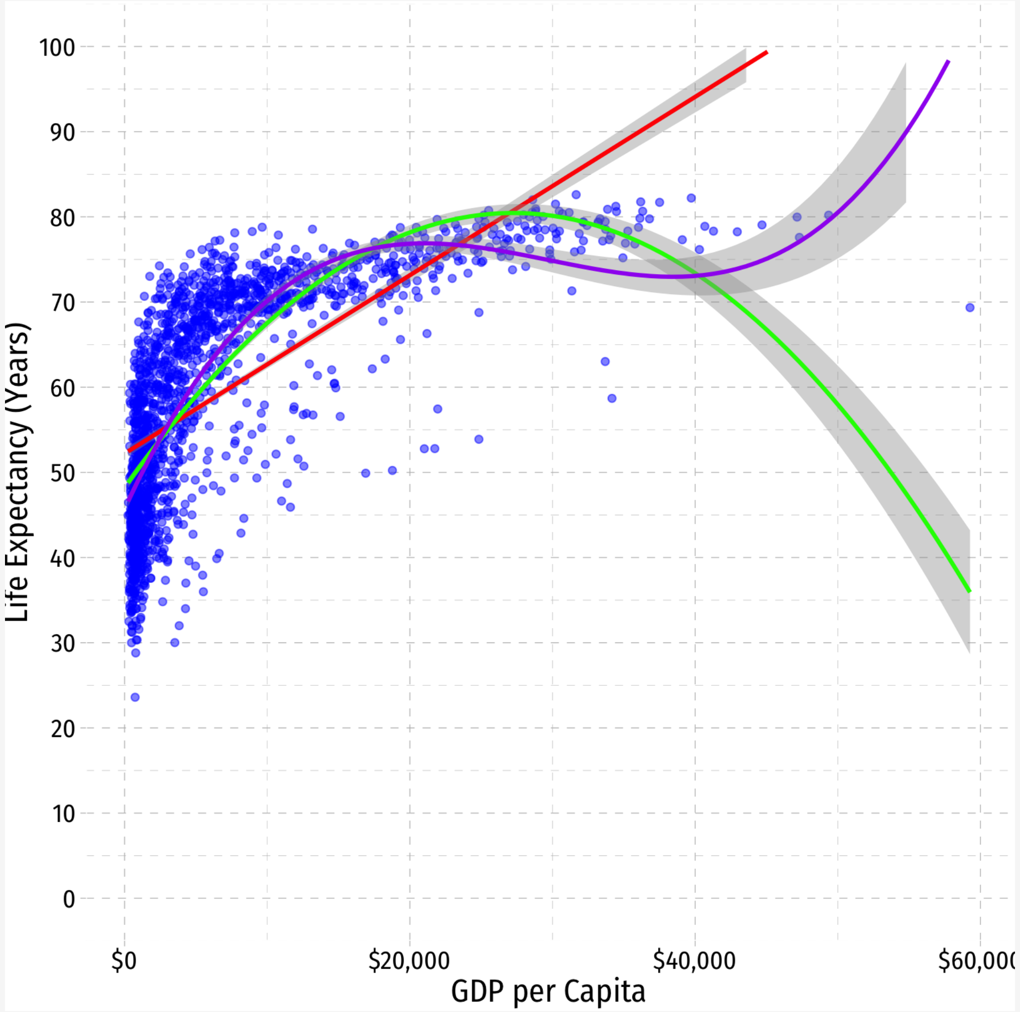
- Determine if $\hat{\beta_2}$ (on X_i^2) is statistically significant:
  - $H_0: \hat{\beta_2}=0$
  - $H_a: \hat{\beta_2} \neq 0$

- Statistically significant $\implies$ we should keep the quadratic model
  - If we only ran a linear model, it would be incorrect!

Determining Polynomials are Necessary II

- Should we keep going up in polynomials?

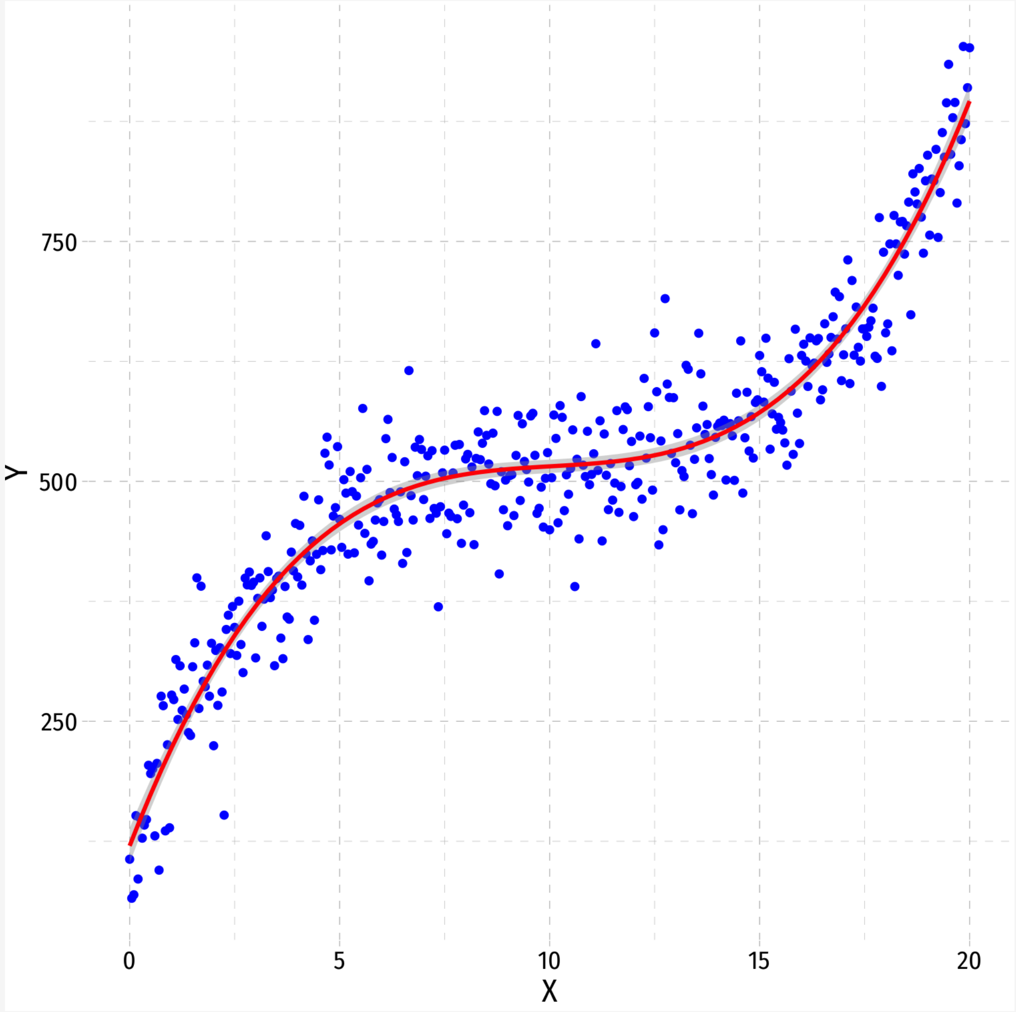
$$\color{#6A5ACD}{\widehat{\text{Life Expectancy}_i} =
\hat{\beta_0}+\hat{\beta_1}
GDP_i+\hat{\beta_2}GDP^2_i+\hat{\beta_3}GDP_i^3}$$



Determining Polynomials are Necessary III

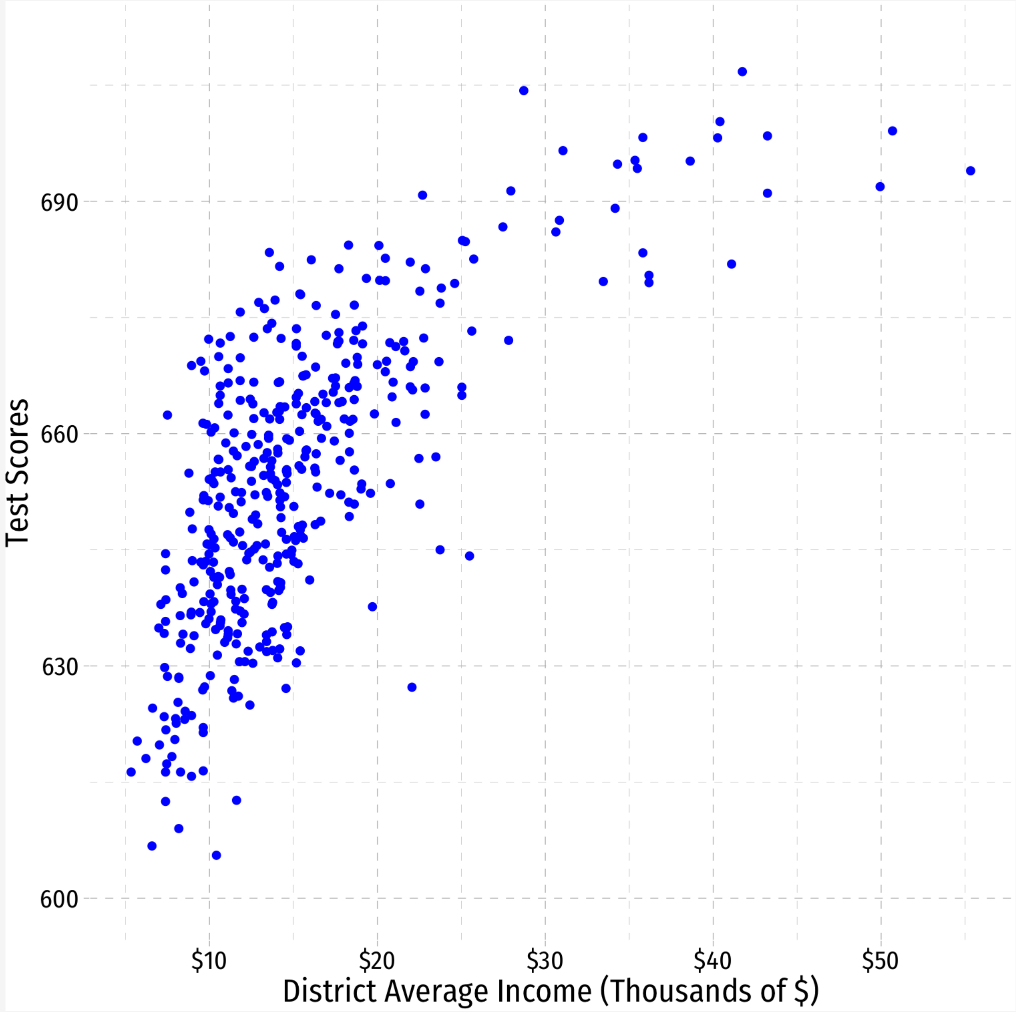
- In general, you should have a **compelling theoretical reason** why data or relationships should **“change direction”** multiple times

- Or clear data patterns that have multiple “bends”



A Second Polynomial Example I

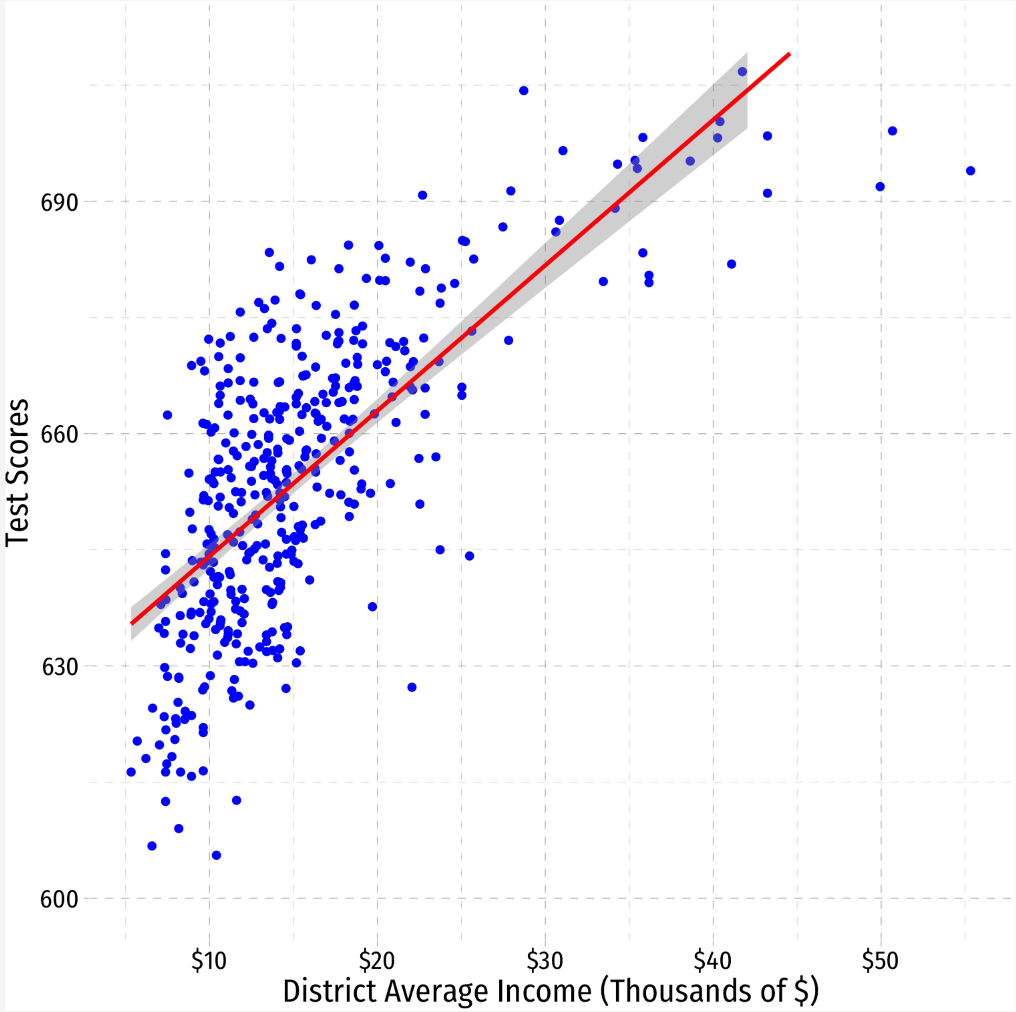
Example: How does a school district's average income affect Test scores?



A Second Polynomial Example I

Example: How does a school district's average income affect Test scores?

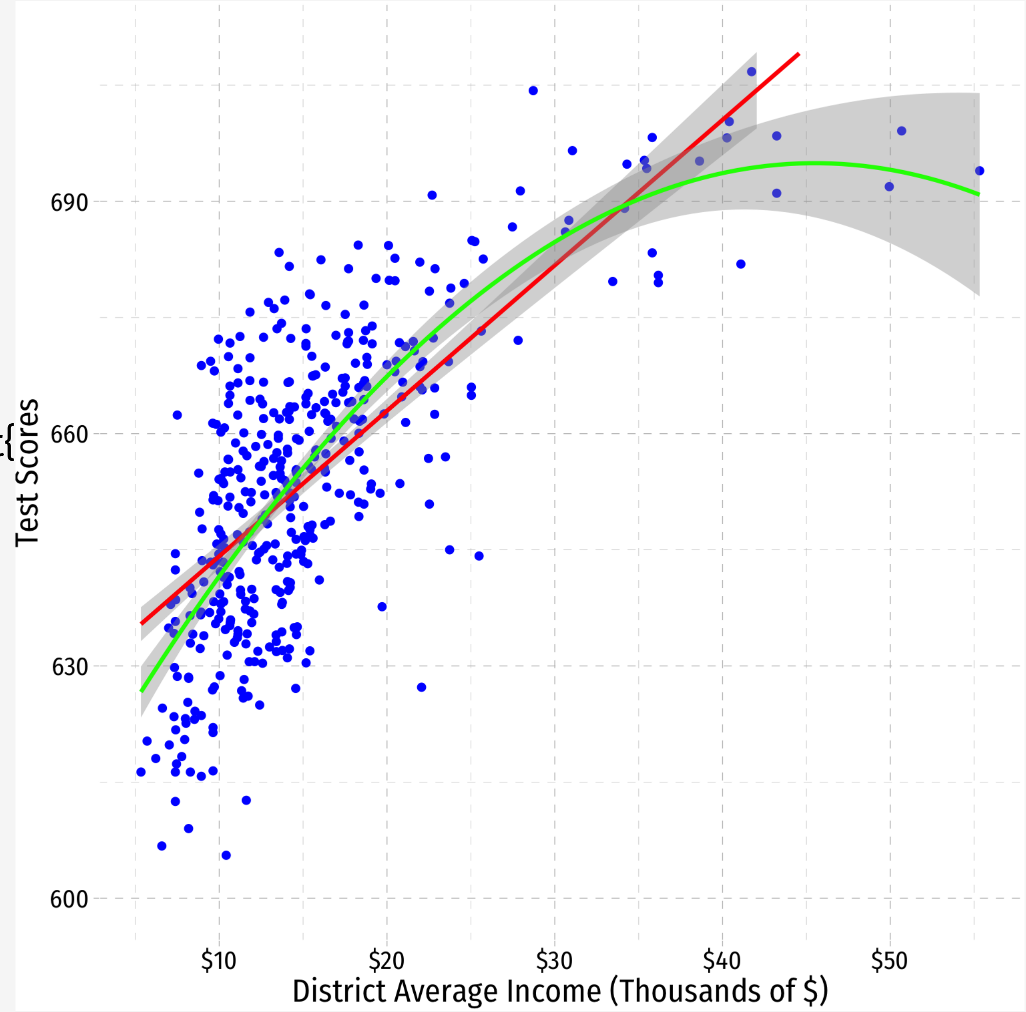
$$\color{red}{\widehat{\text{Test Score}_i}=\hat{\beta_0}+\hat{\beta_1}\text{Income}_i}$$



A Second Polynomial Example I

Example: How does a school district's average income affect Test scores?

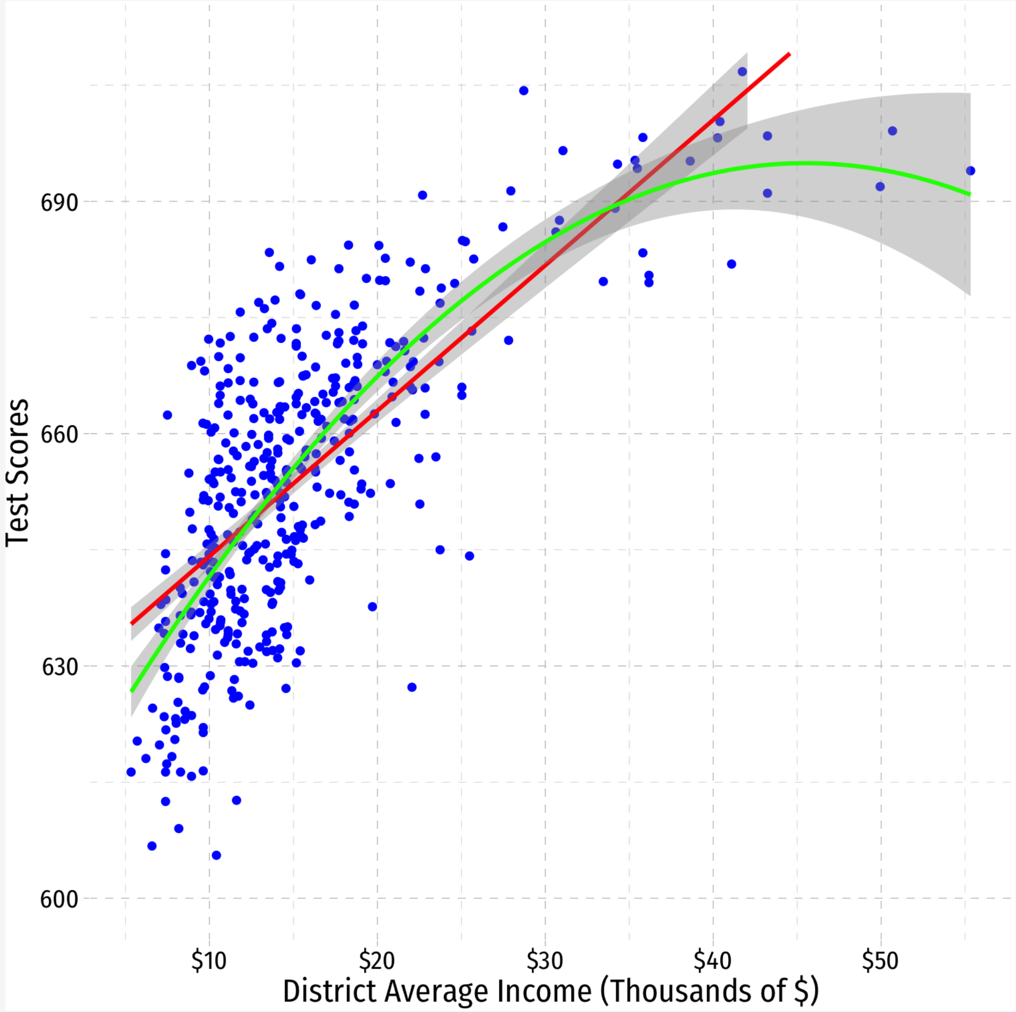
$$\color{green}{\widehat{\text{Test Score}_i}=\hat{\beta_0}+\hat{\beta_1}\text{Income}_i+\hat{\beta_1}\text{



A Second Polynomial Example II

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	607.30173501	3.046219282	199.362449	0.000000e+00
avginc	3.85099474	0.304261693	12.656850	2.690099e-31
I(avginc^2)	-0.04230846	0.006260061	-6.758474	4.713383e-11

3 rows

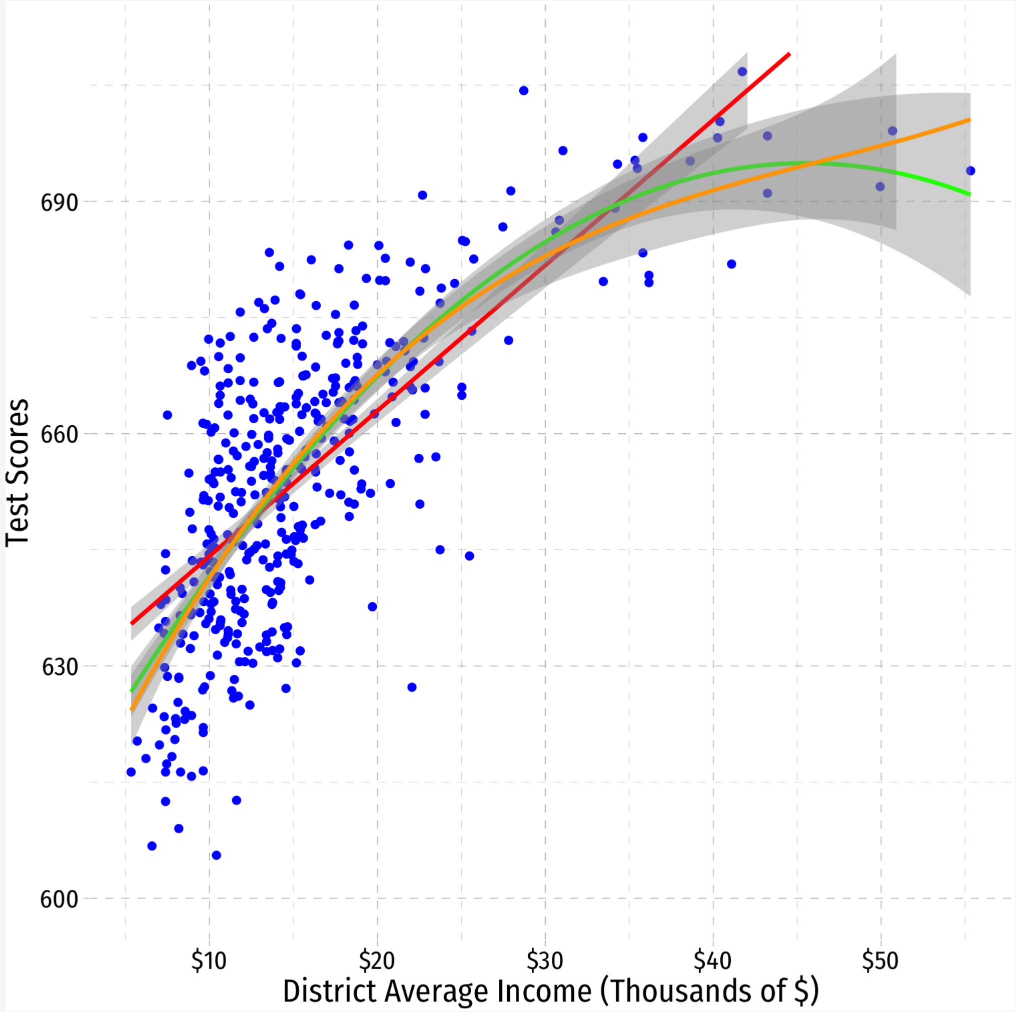


A Second Polynomial Example III

term	estimate	std.error	statistic	p.value
<chr>	<dbl>	<dbl>	<dbl>	<dbl>
(Intercept)	6.000790e+02	5.8295880342	102.936774	4.611745e-298
avginc	5.018677e+00	0.8594537744	5.839379	1.056874e-08
I(avginc^2)	-9.580515e-02	0.0373591998	-2.564433	1.068452e-02
I(avginc^3)	6.854842e-04	0.0004719549	1.452436	1.471343e-01

4 rows

- Should we keep going?



Strategy for Polynomial Model Specification

1. Are there good theoretical reasons for relationships changing (e.g. increasing/decreasing returns)?

2. Plot your data: does a straight line fit well enough?

3. Specify a polynomial function of a higher power (start with 2) and estimate OLS regression

4. Use \(t\)-test to determine if higher-power term is significant

5. Interpret effect of change in \(X\) on \(Y\)

6. Repeat steps 3-5 as necessary